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EXACT STATIONARY PROBABILITY DENSITY FOR SECOND ORDER NON-LINEAR SYSTEMS UNDER EXTERNAL WHITE NOISE EXCITATION

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In this paper exact stationary solutions are constructed for non-linear dynamical systems subjected to stochastic excitation. The solution of the equations (6, 7) in this paper is demonstrated to be unique, and then the results of references [10, 11] are shown to be generalized. Therefore, new exact stationary solutions that satisfies the Fokker–Plank–Kolmogorov (FPK) equation is obtainable.

1. INTRODUCTION

For safety, reliability and economic reasons, the non-linearities of many dynamical engineering systems excited by random disturbances must be taken into account in the design procedures.

In the last 30 years the response of non-linear dynamical systems subjected to stochastic excitation has been extensively studied. In general, very limited classes of non-linear dynamical systems possess exact solutions, and therefore various approximate methods have been proposed for their solutions. Extensive reviews of the subject were published by Crandall and Zhu [1–3], Cai and Lin [4, 5]. It is well known that the response of a dynamical systems to Gaussian white noise excitations is a diffusion process, and the transition probability density of the response process is governed by the FPK equation.

For linear systems the transient probability density function can be obtained by a variety of methods [6]. Exact solutions are usually very difficult to obtain for non-linear stochastic systems. Exact solutions for FPK equations are known only for very special first order non-linear systems [7]. For second and higher order non-linear stochastic systems, solutions have been obtained only for the reduced FPK equations, namely, FPK equations without the time-derivative term. The solutions of a reduced FPK equation is the stationary probability density of the sufficiently long time. Much effort has been made to find the exact stationary solutions of various non-linear stochastic systems. However, only certain classes of second order and a class of higher order non-linear systems under external random excitations have been exactly solved [8–11].

All the exact stationary solutions obtained to date for non-linear stochastic systems under external random excitations are closely related to the Maxwell–Boltzmann distribution which belongs to classical statistical mechanics [12]. The purpose of this paper is to construct the exact stationary probability density of non-linear dynamical systems subjected to external stochastic excitations. In this paper the authors have found that when the FPK equation of the non-linear systems (1) is expressed by the equations (6, 7), only

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two classes of non-linear vibration systems yield exact solutions. In other words, only the system of linear damping and non-linear restoring force, and the systems of non-linear damping and non-linear restoring force, can be expressed by using energy functions possessing exact solutions. Consequently, the authors first showed that the solution of the equations (6, 7) is unique. Then the results of references [10, 11] were generalized on the basis of reference [9], and certain new exact stationary probability density of general FPK equations was constructed.

2. UNIQUENESS OF EXACT STATIONARY SOLUTIONS OF CLASSICAL NON-LINEAR DYNAMICAL SYSTEMS

Consider the following general non-linear system

$$\ddot{x} + g(x, \dot{x}) = w(t), \tag{1}$$

where $w(t)$ is a zero-mean Gaussian white noise with the delta-type correlation functions $E[w(t)w(t+\tau)] = 2\pi\phi\delta(\tau)$. The stationary probability density $p(y_1, y_2)$ of the system response is governed by the reduced Fokker–Planck equation [13]:

$$-y_2\,\partial p/\partial y_1 + \partial[g(y_1, y_2)p]/\partial y_2 + \pi\phi\,\partial^2 p/\partial y_2^2 = 0, \tag{2}$$

where y_1 and y_2 are the state variables of $y_1 = x(t)$ and $y_2 = \dot{x}(t)$, respectively.
Let

$$p(y_1, y_2) = c\exp[-(1/\pi\phi)f(y_1, y_2)], \tag{3}$$

where c is a normalization constant, and $f(y_1, y_2)$ is an arbitrary non-linear function. Of course, expression (3) must be non-negative and normalizable for $p(y_1, y_2)$ to be a valid probability density. Substituting equation (3) into equation (2) yields $p(y_1, y_2)$ as a valid probability density. Substituting equation (3) into equation (2) yields

$$p(y_2(f_{y_1}/\pi\phi) - g(f_{y_2}/\pi\phi) + f_{y_2}^2/\pi\phi + g_{y_2} - f_{y_2y_2}) = 0, \tag{4}$$

where

$$f_{y_1} = \partial f/\partial y_1, \qquad f_{y_2} = \partial f/\partial y_2, \qquad f_{y_2y_2} = \partial^2 f/\partial y_2^2.$$

Because $p(y_1, y_2) \neq 0$, equation (4) and the following equation are equivalent:

$$\partial g/\partial y_2 - f_{y_2}g/\pi\phi + y_2 f_{y_1}/\pi\phi + f_{y_2}^2/\pi\phi - f_{y_2y_2} = 0 \tag{5}$$

There has not previously been a general method for solving equation (5). First we use the following method:
Let

$$f_{y_2y_2}(y_1, y_2) - g_{y_2}(y_1, y_2) = 0, \tag{6}$$

$$y_2 f_{y_1}(y_1, y_2) - g(y_1, y_2)f_{y_2}(y_1, y_2) + f_{y_2}^2(y_1, y_2) = 0. \tag{7}$$

If the solution of the above equations can satisfy equations (6, 7), then the solution must satisfy the FPK equation (5).

One obtains from equation (7)

$$g(y_1, y_2) = f_{y_2}(y_1, y_2) + y_2 f_{y_1}(y_1, y_2)/f_{y_2}(y_1, y_2). \tag{8}$$

Deriving partial derivation with respect to y_2 in Equation (8) yields

$$g_{y_2}(y_1, y_2) = f_{y_2 y_2}(y_1, y_2) + \frac{f_{y_1}(y_1, y_2)}{f_{y_2}(y_1, y_2)} + y_2 \frac{\partial}{\partial y_2}\left(\frac{f_{y1}(y_1, y_2)}{f_{y_2}(y_1, y_2)}\right). \tag{9}$$

Substituting equation (9) into equation (6) yields

$$y_2 \frac{\partial}{\partial y_2}\left[\frac{f_{y_1}(y_1, y_2)}{f_{y_2}(y_1, y_2)}\right] + \frac{f_{y_1}(y_1, y_2)}{f_{y_2}(y_1, y_2)} = 0. \tag{10}$$

The general solution of equation (10) is

$$f_{y_1}(y_1, y_2)/ f_{y_2}(y_1, y_2) = r(y_1)/y_2. \tag{11}$$

where $r(y_1)$ is an arbitrary function of y_1.
According to the theory of differential equation, if

$$r(y_1)\, \mathrm{d}y_1 + y_2\, \mathrm{d}y_2 = 0, \tag{12}$$

then the sufficient and necessary condition of the first integration $H(y_1, y_2) = C$ for equation (12) is

$$\partial H(y_1, y_2)/\partial y_1 - (r(y_1)/y_2)\, \partial H(y_1, y_2)/\partial y_2 = 0. \tag{13}$$

By comparing (13) and (11) one obtains

$$f(y_1, y_2) = Q(H). \tag{14}$$

This is a general solution to equation (11) where $Q(H)$ is an arbitrary differentiable function of H.

One can readily solve the first integration from equation (12) as follows:

$$\frac{y_2^2}{2} + \int_0^{y_1} r(y_1)\, \mathrm{d}y_1 = C, \tag{15}$$

setting

$$H(y_1, y_2) = \frac{y_2^2}{2} + \int_0^{y_1} r(y_1)\, \mathrm{d}y_1. \tag{16}$$

Substituting equation (14) into equation (8) yields

$$g(y_1, y_2) = r(y_1) + (\mathrm{d}Q(H)/\mathrm{d}H) y_2 \tag{17}$$

It follows that $g(y_1, y_2)$ which satisfies equations (6, 7) can only be the form of equation (17), namely, the classical non-linear system of reference [13]

$$\ddot{x} + [\mathrm{d}Q(H)/\mathrm{d}H]\dot{x} + r(x) = w(t), \tag{18}$$

whose exact stationary probability density function is [13]:

$$p(x, \dot{x}) = c \exp(-(1/\pi\phi)Q(H)) \tag{19}$$

So far, the solution of equations (6, 7) have been demonstrated to be unique.

3. GENERALIZED METHOD

By introducing a function $h(y_1, y_2)$, which satisfies the following equations

$$f_{y_2 y_2}(y_1, y_2) - g_{y_2}(y_1, y_2) = (1/\pi\phi)h(y_1, y_2), \tag{20}$$

$$y_2 f_{y_1}(y_1, y_2) - g(y_1, y_2) f_{y_2}(y_1, y_2) + f^2_{y_2}(y_1, y_2) = h(y_1, y_2), \tag{21}$$

then $h(y_1, y_2)$ must satisfy the FPK equation (5).

One can solve from equation (20)

$$f_{y_2}(y_1, y_2) = g(y_1, y_2) - g(y_1, 0) + f_{y_2}(y_1, 0) + \frac{1}{\pi\phi}\int_0^{y_2} h(y_1, y_2)\,\mathrm{d}y_2, \tag{22}$$

$$f(y_1, y_2) = \int_0^{y_2} g\,\mathrm{d}y_2 - y_2 g(y_1, 0) + y_2 f_{y_2}(y_1, 0) + f(y_1, 0) + \frac{1}{\pi\phi}\int_0^{y_2}\int_0^{y_2} h(y_1, y_2)\,\mathrm{d}y_2\,\mathrm{d}y_2, \tag{23}$$

$$f_{y_1}(y_1, y_2) = \int_0^{y_2} g_{y_1}\,\mathrm{d}y_2 - y_2 g_{y_1}(y_1, 0) + y_2 f_{y_2 y_1}(y_1, 0) + f(y_1, 0) + \frac{1}{\pi\phi}\int_0^{y_2}\int_0^{y_2} h_{y_1}(y_1, y_2)\,\mathrm{d}y_2\,\mathrm{d}y_2. \tag{24}$$

Substituting equations (22) and (24) into equation (21) yields

$$y_2\left[\int_0^{y_2} g_{y_1}\,\mathrm{d}y_2 - y_2 g_{y_1}(y_1, 0) + y_2 f_{y_2 y_1}(y_1, 0) + f_{y_1}(y_1, 0) + \frac{1}{\pi\phi}\int_0^{y_2}\int_0^{y_2} h_{y_1}(y_1, y_2)\,\mathrm{d}y_2\,\mathrm{d}y_2\right]$$
$$-\left[g(y_1, 0) - f_{y_2}(y_1, 0) - \frac{1}{\pi\phi}\int_0^{y_2} h(y_1, y_2)\,\mathrm{d}y_2\right]$$
$$\times\left[g(y_1, y_2) - g(y_1, 0) + f_{y_2}(y_1, 0) + \frac{1}{\pi\phi}\int_0^{y_2} h(y_1, y_2)\,\mathrm{d}y_2\right] = h(y_1, y_2). \tag{25}$$

To simplify calculation, one expresses equation (25) as

$$y_2 f_{y_1}(y_1, 0) = g(y_1, 0)[g(y_1, y_2) - g(y_1, 0)] + y_2^2 g_{y_1}(y_1, 0) - y_2\int_0^{y_2} g_{y_1}(y_1, y_2)\,\mathrm{d}y_2 + h_1(y_1, y_2), \tag{26}$$

where

$$h_1(y_1 y_2) = h(y_1, y_2) - \frac{y_2}{\pi\phi}\int_0^{y_2}\int_0^{y_2} h_{y_1}(y_1, y_2)\,\mathrm{d}y_2\,\mathrm{d}y_2$$
$$-\frac{1}{\pi\phi}\int_0^{y_2} h(y_1, y_2)\,\mathrm{d}y_2[g(y_1, y_2) - 2g(y_1, 0) + \frac{1}{\pi\phi}\int_0^{y_2} h(y_1, y_2)\,\mathrm{d}y_2]$$

$$-f_{y_2}(y_1,0)\left[g(y_1,y_2)-2g(y_1,0)+\frac{2}{\pi\phi}\int_0^{y_2}h(y_1,y_2)\,\mathrm{d}y_2+f_{y_2}(y_1,0)\right]$$

$$-y_2^2 f_{y_2y_1}(y_1,0). \tag{27}$$

It is assumed, that the undetermined function

$$h(y_1,y_2)=y_2h_2(y_1)+h_3(y_1). \tag{28}$$

Substituting equation (28) into equation (27) yields

$$h_1(y_1,y_2)=h_3(y_1)-f_{y_2}(y_1,0)[g(y_1,y_2)-2g(y_1,0)+f_{y_2}(y_1,0)]+y_2\Bigg[h_2(y_1)$$

$$-\frac{g(y_1,y_2)-2g(y_1,0)+2f_{y_2}(y_1,0)}{\pi\phi}h_3(y_1)\Bigg]-y_2^2\Bigg[\frac{h_3^2(y_1)}{(\pi\phi)^2}$$

$$+\frac{(g(y_1,y_2)-2g(y_1,0)+2f_{y_2}(y_1,0))h_2(y_1)}{2\pi\phi}+f_{y_2y_1}(y_1,0)\Bigg]$$

$$-y_2^3\left[\frac{h_2(y_1)h_3(y_1)}{(\pi\phi)^2}+\frac{h_3'(y_1)}{2\pi\phi}\right]-y_2^4\left[\frac{h_2^2(y_1)}{4(\pi\phi)^2}+\frac{h_2'(y_1)}{6\pi\phi}\right]. \tag{29}$$

Since $f_{y_1}(y_1,0)$ is only a function of y_1 the right side of equation (26) must be independent of y_2. Therefore, one can solve the undetermined function $h_1(y_1,y_2)$ from the restricted condition. Substituting equation (26) into equation (23), one can determine the stationary probability density $f(y_1,y_2)$.

3.1. EXAMPLE 1

Consider the following non-linear system

$$\ddot{x}+a_0x+a_2x\dot{x}^2+a_0a_1\dot{x}/(a_0+a_2\dot{x}^2)=w(t). \tag{30}$$

Thus

$$g(x,\dot{x})=a_0x+a_2x\dot{x}^2+a_0a_1\dot{x}/(a_0+a_2\dot{x}^2). \tag{31}$$

Substituting equation (31) into equation (26, 29) and setting $y_1=x$ as well as $y_2=\dot{x}$ yields

$$y_2f_{y_1}(y_1,0)=[h_3(y_1)+a_0y_1-f_{y_2}(y_1,0)]f_{y_2}(y_1,0)+y_2\Bigg[h_2(y_1)\left(1-\frac{a_0a_1}{2\pi\phi a_2}\right)$$

$$+\frac{a_0y_1h_3(y_1)}{\pi\phi}\frac{a_0a_1\left(a_0y_1+\dfrac{a_0h_2(y_1)}{2\pi\phi a_2}-f_{y_2}(y_1,0)-h_3(y_1)y_2\right)}{a_0+a_2y_2^2}\Bigg]$$

$$+y_2^2\left[a_0a_2y_1^2+\frac{a_0y_1h_2(y_1)}{2\pi\phi}-\frac{h_3^2(y_1)}{(\pi\phi)^2}a_2y_1f_{y_2}(y_1,0)-f_{y_2y_1}(y_1,0)\right]$$

$$- y_2^3\left[\frac{h_2(y_1)h_3(y_1)}{(\pi\phi)^2} + \frac{h_3'(y_1)}{2\pi\phi} + \frac{a_2 y_1 h_3(y_1)}{\pi\phi}\right]$$

$$- y_2^4\left[\frac{a_2}{3} + \frac{a_2 y_1 h_2(y_1)}{2\pi\phi} + \frac{h_2'(y_1)}{6\pi\phi} + \frac{h_2^2(y_1)}{4(\pi\phi)^2}\right]. \tag{32}$$

Since the right side of equation (32) is independent of y_2, the restricted conditions are

$$h_3(y_1) = 0, \qquad f_{y_2}(y_1, 0) = 0, \qquad y_1 + h_2(y_1)/2\pi\phi a_2 = 0, \tag{33–35}$$

$$a_2/3 + a_2 y_1 h_2(y_1)/2\pi\phi + h_2'(y_1)/6\pi\phi + h_2^2(y_1)/4(\pi\phi)^2 = 0, \tag{36}$$

the unique solution can be obtained from the equations (33–36) as

$$h_2(y_1) = -2\pi\phi a_2 y_1. \tag{37}$$

Substituting equation (37) into equation (32) yields

$$f_{y_1}(y_1, 0) = y_1(a_0 a_1 - 2\pi\phi a_2). \tag{38}$$

$f(y_1, 0)$ is solved from equation (38) as

$$f(y_1, 0) = (y_1^2/2)(a_0 a_1 - 2\pi\phi a_2). \tag{39}$$

Substituting equation (39) into equation (23) yields

$$f(y_1, y_2) = (a_0 a_1/2a_2)\ln(a_0 + a_2 y_2^2/a_0) + y_1^2(a_0 a_1/2 - \pi\phi a_2). \tag{40}$$

Thus the exact stationary joint probability density of the non-linear system defined by equation (30) is obtained

$$p(x, \dot{x}) = c(1 + a_2\dot{x}^2/a_0)^{-a_0 a_1/2\pi\phi a_2}\exp[-(1/\pi\phi)(a_0 a_1/2 - \pi\phi a_2)x^2]. \tag{41}$$

The astringent conditions of the above probability density are

$$a_0 a_1/2 - \pi\phi a_2 > 0, \qquad a_0 a_1/a_2 > 0 \tag{42}$$

3.2. EXAMPLE 2

Consider the following non-linear system

$$\ddot{x} + (l/2)x(1 + x^4) + 4\pi\phi x/(1 + x^4) + (1 + 2l)x^2\dot{x} + \dot{x}^2/x = w(t). \tag{43}$$

Thus

$$g(x, \dot{x}) = (l/2)x(1 + x^4) + 4\pi\phi x/(1 + x^4) + (1 + 2l)x^2\dot{x} + \dot{x}^2/x. \tag{44}$$

Substituting equation (44) into equation (26, 29) yields

$$y_2 f_{y_1}(y_1, 0) = h_3(y_1) - \frac{h_3^2(y_1)}{(\pi\phi)^2} + \left[\frac{l y_1(1 + y_1^4)}{2} + \frac{4\pi\phi y_1}{1 + y_1^4}\right] f_{y_2}(y_1, 0)$$

$$+ y_2\Bigg\{(1 + 2l)y_1^3\left[\frac{l(1 + y_1^4)}{2} + \frac{4\pi\phi}{1 + y_1^4}\right] + h_2(y_1) - (1 + 2l)y_1^2 f_{y_2}(y_1, 0)$$

$$+ \frac{h_3(y_1)}{\pi\phi}\left[\frac{l y_1(1 + y_1^4)}{2} + \frac{4\pi\phi y_1}{1 + y_1^4} - 2 f_{y_2}(y_1, 0)\right]\Bigg\}$$

$$-y_2^2\left\{\frac{h_3^2(y_1)}{(\pi\phi)^2}+\frac{(1+2l)y_1^2h_3(y_1)}{\pi\phi}-\frac{h_2(y_1)}{2\pi\phi}\left[\frac{ly_1(1+y_1^4)}{2}+\frac{4\pi\phi y_1}{1+y_1^4}-2f_{y_2}(y_1,0)\right]\right.$$

$$\left.+f_{y_2y_1}(y_1,0)-\frac{l(1+y_1^4)}{2}-\frac{4\pi\phi}{1+y_1^4}+\frac{f_{y_1}(y_1,0)}{y_1}\right\}$$

$$-y_2^3\left\{\frac{h_3'(y_1)}{2\pi\phi}+\frac{h_3(y_1)}{\pi\phi}\left[\frac{1}{y}+\frac{h_2(y_1)}{\pi\phi}\right]+\frac{1+2l}{2\pi\phi}y_1^2h_2(y_1)+(1+2l)y_1\right\}$$

$$-y_2^4\left[\frac{h_2'(y_1)}{6\pi\phi}+\frac{h_2^2(y_1)}{4(\pi\phi)^2}+\frac{h_2(y_1)}{2\pi\phi y_1}-\frac{1}{3y_1^2}\right]. \tag{45}$$

The restricted conditions of the above equation are

$$h_2(y_1)=-2\pi\phi/y_1,\qquad h_3(y_1)=-2\pi\phi l y_1^2,\qquad f_{y_2}(y_1,0)=ly_1(1+y_1^4)/2. \tag{46–48}$$

Substituting equations (46–48) into equation (45) yields

$$f_{y_1}(y_1,0)=4\pi\phi y_1^3/(1+y_1^4)+l^2y_1^3(1+y_1^4)-2\pi\phi/y_1. \tag{49}$$

$f(y_1,0)$ is solved from equation (49) as

$$f(y_1,0)=\pi\phi\ \ln(1+y_1^4/y_1^2)+l^2y_1^4(2+y_1^4)/8. \tag{50}$$

Substituting equation (50) into equation (23) yields

$$f(y_1,y_2)=\pi\phi\ \ln(1+y_1^4)/y_1^2+y_1^2y_2^2/2+ly_1y_2(1+y_1^4)/2+l^2y_1^4(2+y_1^4)/8. \tag{51}$$

The exact probability density of equation (43) is given as

$$p(x,\dot{x})=[Ax^2/(1+x^4)]\exp\{-(x^2/2\pi\phi)[\dot{x}+l(1+x^4)/2x]^2\}. \tag{52}$$

The above probability density satisfies equation (5) in the two regions $(-\infty,0)$ and $(0,+\infty)$.

3.3. EXAMPLE 3

$$\ddot{x}+(l_1-x\dot{x}^2)/(a^2+x^2/2)+l_2\dot{x}/(a^2+x^2/2)^2=w(t). \tag{53}$$

Thus

$$g(x,\dot{x})=(l_1-x\dot{x}^2)/(a^2+x^2/2)+l_2\dot{x}/(a^2+x^2/2)^2. \tag{54}$$

Substituting equation (54) into equation (26, 29) yields

$$y_2f_{y_1}(y_1,0)=y_2\left[h_2(y_1)+\frac{l_1h_3(y_1)}{\pi\phi(a^2+y_1^2/2)}-\frac{l_2f_{y_2}(y_1,0)}{(a^2+y_1^2/2)^2}\right.$$

$$\left.+\frac{l_1l_2}{(a^2+y_1^2/2)^3}-\frac{2h_3(y_1)f_{y_2}(y_1,0)}{\pi\phi}\right]+f_{y_2}^2(y_1,0)-h_3(y_1)$$

$$-\frac{l_1f_{y_2}(y_1,0)}{a^2+y_1^2/2}-y_2^2\left[\frac{h_3^2(y_1)}{(\pi\phi)^2}+f_{y_2y_1}(y_1,0)+\frac{h_2(y_1)f_{y_1}(y_1,0)}{\pi\phi}\right.$$

$$- \frac{y_1 f_{y_2}(y_1, 0)}{a^2 + y_1^2/2} - \frac{l_1 h_2(y_1)}{2\pi\phi(a^2 + y_1^2/2)} + \frac{l_1 y_1}{(a^2 + y_1^2/2)^2} + \frac{l_2 h_3(y_1)}{\pi\phi(a^2 + y_1^2/2)^2}\Bigg]$$

$$- y_2^3\left[\frac{h_3'(y_1)}{2\pi\phi} + \frac{h_2(y_1)h_3(y_1)}{(\pi\phi)^2} - \frac{y_1 h_3(y_1)}{\pi\phi(a^2 + y_1^2/2)}\right.$$

$$\left. + \frac{l_2 h_2(y_1)}{2\pi\phi(a^2 + y_1^2/2)^2} - \frac{l_2 y_1}{(a^2 + y_1^2/2)^3}\right] - y_2^4\left[\frac{h_2'(y_1)}{6\pi\phi} + \frac{h_2^2(y_1)}{4(\pi\phi)^2}\right.$$

$$\left. - \frac{y_1 h_2(y_1)}{2\pi\phi(a^2 + y_1^2/2)} - \frac{(a^2 - y_1^2/2)}{3(a^2 + y_1^2/2)^2}\right]. \tag{55}$$

The restricted conditions of equation (55) are

$$h_2(y_1) = 2\pi\phi y_1/(a^2 + y_1^2/2), \qquad h_3(y_1) = 0, \qquad f_{y_2}(y_1, 0) = l_1/(a^2 + y_1^2/2). \tag{56–58}$$

Substituting equations (56–58) into equation (55) yields

$$f_{y_1}(y_1, 0) = 2\pi\phi y_1/(a^2 + y_1^2/2). \tag{59}$$

The solution of equation (59) is given by

$$f(y_1, 0) = 2\pi\phi \ln(a^2 + y_1^2/2) \tag{60}$$

Substituting equation (60) into equation (23) yields

$$f(y_1, y_2) = \frac{l_2 y_2^2}{2(a^2 + y_1^2/2)^2} + \frac{l_1 y_2}{(a^2 + y_1^2/2)} + 2\pi\phi \ln(a^2 + y_1^2/2) \tag{61}$$

Thus the exact stationary joint probability density of the non-linear system defined by equation (53) is obtained as

$$p(x, \dot{x}) = \frac{A}{(a^2 + x^2/2)^2} \exp\left\{-\frac{l_2}{2\pi\phi(a^2 + x^2/2)^2}\left(\dot{x} + \frac{l_1(a^2 + x^2/2)}{l_2}\right)^2\right\} \tag{62}$$

4. CONCLUSIONS

The solution of the equations (6, 7) has been demonstrated to be unique. In other words, only two classes of non-linear dynamical systems possess exact stationary solutions. These are recently published results [9, 13]. The system of linear damping and a non-linear restoring force is only a special case of the system of non-linear damping and a non-linear restoring force. Therefore, the results obtained in this paper encouraged the authors to investigate other methods, loading new exact stationary solutions which satisfy the FPK equation (5). The important idea of this paper is that an undetermined function $h(x, \dot{x})$ which satisfies the FPK equation can be found. The idea is an important one for engineering. A model has been provided for finding other classes of new exact solutions of non-linear systems when using general methods.

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